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Analysis of Free-Piston Stirling Engine/Linear Alternator Systems Part 2: Results

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A coupled electric, dynamic, and thermodynamic analysis, developed by the authors for free-piston Stirling engine/linear alternator systems and reported in the first part of this study, is applied here to an existing and operating plant in order to satisfy the following objectives: firstly, to test its capability to predict the performance of such devices both at design and at off-design conditions, by means of a comparison with experimental results. This is very important since a high prediction capability allows the developed analysis to be used not only for performance evaluation, but also for design purposes. Secondly, to quantify separately the effect of the various elements that have a stabilizing influence on the system when the load is changing. In this way it is possible to single out those parameters that may be varied, when designing the system, in order to improve its stability. In fact, a good stability is required when large changes of the operating conditions are foreseen during the operation. In order to illustrate these features, we have considered, as an example of a thermoelectric dynamic energy conversion system, the well-known Harwell thermo-mechanical generator, whose testing has been carried out, and the results provided, by Cooke-Yarborough for use by authors.

Nomenclature

A = cross-sectional area
B = flux density in the permanent magnet
c_p, c_v = specific heats at constant pressure and volume
D' = damping coefficient per unit of mass
F = force
f = frequency, Hz
I_g = amplitude of the induced electric current i_g
i_g = induced electric current i_g
k_g = alternator constant
M = moving element mass
m = gas mass
$\dot{m}$ = gas mass flow rate
N = total number of turns in the alternator coils
P_{el} = mean electric power delivered by thermo-mechanical generator
p = pressure
Q = heat exchanged
R = specific gas constant
R_l = load resistance
r = displacer–piston stroke ratio, X_d/X_p
S_a = stiffness of the alternator springs
S_c = stiffness of the mounting springs
S_{dc} = stiffness of the cantilever spring
S_{pc} = stiffness of the diaphragm
S' = stiffness coefficient per unit of mass
T = temperature
t = time
V = volume
w = width of each pole-piece gap at midstroke
X = moving element stroke
x = displacement
$\dot{x}$ = velocity
$\ddot{x}$ = acceleration
α_{dc} = displacer–casing mass ratio, M_d/M_c
α_{pc} = piston–casing mass ratio, M_p/M_c
γ = ratio of specific heats, c_p/c_v
Δp = pressure drop
μ = polytropic process exponent
ϕ = piston–displacer phase angle
ϕ_c = casing–displacer phase angle
ω = angular frequency, $2\pi f$ rad/s

Subscripts
a–sl = linear alternator–static load subsystem
b = bounce space
c = casing, compression space
c/rg = compression space–regenerator interface
d = displacer
e = expansion space
h = engine hot-end
k = engine cold-end
m = mean value
ms = mechanical springs
p = piston, diaphragm hub
pm = permanent magnet
rg = regenerator
rg/e = regenerator–expansion space interface
w = working gas circuit

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Introduction

THE analysis proposed by the authors for free-piston Stirling engine (FPSE)/linear alternator–static load (LA–sL) systems[1] can be applied also to thermo-mechanical generator (TMG) systems,[2] although there are some differences (not basic) relative to the dynamic, thermodynamic, and electric behavior of these devices.

The TMG is in effect a single-cylinder FPSE with displacer sprung to ground, though it does not have a piston. It has instead a metal diaphragm with a rigid hub, which deflects

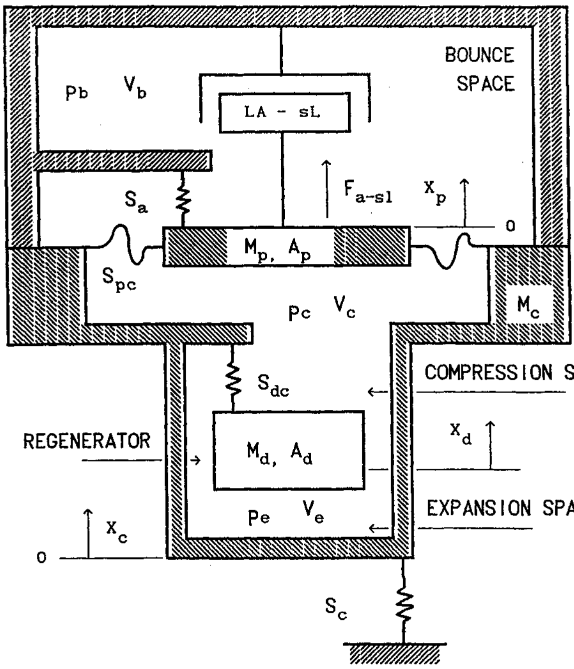

Fig. 1 TMG scheme.

under gas pressure, driving a short-stroke linear alternator of variable-gap flux-switching type. The displacer oscillates on a spring, as is indicated in Fig. 1, with a clearance between it and the cylinder wall that works as a regenerator.

The motion of the displacer is not a direct result of the pressure forces (there is no gas spring, which is replaced by a mechanical spring), but it is a result of feedback from the casing motion. In other words, the casing motion drives the displacer through a mechanical spring having S_{dc} as a stiffness (see Fig. 1). Therefore, it is not possible to neglect the casing motion, as is usually done in the dynamic analyses of FPSEs,[3] although its amplitude is considerably smaller than both the diaphragm and displacer amplitudes. Therefore, the TMG has to be considered, from a dynamic point of view, as a system having three degrees of freedom. However, a simplifying assumption, not far from the TMG actual behavior, allows it to be analytically described as a dynamic system having two degrees of freedom. In this way most of the results obtained by the analysis proposed in Ref. 1 for FPSE/LA-sL systems are still valid.

As far as the thermodynamic behavior is concerned, the TMG does not have a heater and cooler, so that the expansion and compression spaces have to exchange not only work, but also heat with the external sources. Consequently, the adiabatic model,[4] which usually leads to quite satisfactory results in the Stirling engine thermodynamic performance prediction, cannot be applied to the TMG working gas circuit. Instead, the classical isothermal model,[4] can be applied, although it usually does not lead to completely satisfactory results. Then, in this study, a so-called polytropic model has been introduced and compared with the isothermal one.

Both models have been linked with the electric and dynamic analysis, used to calculate the effect of the various stabilizing elements and to evaluate the theoretical curves of P_{el}, ϕ, piston amplitude $X_p/2$, displacer amplitude $X_d/2$, f, I_g, as functions of R_l. These curves have been compared with the experimental ones obtained for the same engine by Cooke-Yarborough.[5]

In order to obtain, for the TMG, the algebraic expressions of the stiffness S' and damping D' coefficients, a specific dynamic and thermodynamic analysis has been developed, as reported in the following sections.

TMG Dynamic Analysis

Referring to Fig. 1, the equations of motion for piston, displacer, and casing are, respectively,

$$M_p\ddot{x}_p = (p_c - p_b)A_p - (S_{pc} + S_a)(x_p - x_c) + F_{a-sl} \quad (1)$$

$$M_d\ddot{x}_d = (p_e - p_c)A_d - S_{dc}(x_d - x_c) \quad (2)$$

$$M_c\ddot{x}_c = (p_b - p_c)A_p + (p_c - p_e)A_d - F_{a-sl} + (S_{pc} + S_a)(x_p - x_c) - S_c x_c + S_{dc}(x_d - x_c) \quad (3)$$

where p_e and p_c have been assumed as follows:

$$p_e = p_w + \Delta p_w/2 \qquad p_c = p_w - \Delta p_w/2 \quad (4)$$

The term Δp_w represents the pressure drop between the expansion and compression spaces of the working gas circuit, and is due to the gas flow through the regenerator and into or out of the expansion and compression spaces. However, p_w is the working gas circuit pressure, evaluated neglecting the pressure loss Δp_w.

The assumption of Eqs. (4), physically consistent only with the following expression of the volumetric flow rate $\dot{Q}$ through the regenerator:

$$\dot{Q} = \tfrac{1}{2}[A_p\dot{x}_p - 2A_d\dot{x}_d + (2A_d - A_p)\dot{x}_c]$$

which may be obtained by applying the same procedure used for an FPSE in Ref. 6, Chap. 3, allows an adequate representation of the influence of Δp_w on the laws of motion of the reciprocating elements.

If the TMG is working in some steady-state operation, which depends on the adopted operating conditions, the centers of the piston, displacer, and casing strokes do not change while the engine is working, as is explained in the subsection of Appendix A. Thus, the equations of motion (1–3) have been written assuming, as a frame of reference for each oscillating part (piston, displacer, and casing), the own midpoint of each respective stroke. Therefore, the adopted frame of reference is inertial.

The following functional dependences are valid:

$$p_w = p_w(x_p, x_d, x_c)$$

$$p_b = p_b(x_p, x_c)$$

$$\Delta p_w = \Delta p_w(\dot{x}_p, \dot{x}_d, \dot{x}_c)$$

since the casing motion must be taken into account.

The force F_{a-sl} exerted by the LA–sL subsystem, which appears in Eqs. (1) and (3), is given as shown in Ref. 1, by the sum of an electromagnetic force opposed to the piston displacement:

$$F_{a-sl,1} = k_g i_g \quad (5)$$

where the term k_g is given in Appendix A, and a magnetic force equivalent to a negative stiffness

$$F_{a-sl,2} = -S_{a-sl,2}(x_p - x_c) \quad (6)$$

with $S_{a-sl,2} < 0$. Since the positive stiffness S_{pc} provided by the diaphragm is not able to balance this negative stiffness, S_{pc} is augmented[7] by strong centering springs having S_a as a positive stiffness (see Fig. 1), in such a way that $S_a = -S_{a-sl,2}$.

The i_g, which appears in Eq. (5), is the current flowing through the windings of the linear alternator[1] used in TMG. It is given, considering also the casing motion, by

$$i_g = -a'_{1,1}(x_p - x_c) - a'_{2,1}(\dot{x}_p - \dot{x}_c) \quad (7)$$

where the $a'_{1,1}$ and $a'_{2,1}$ coefficients are given in Ref. 1, in the case of a linear alternator connected to a static load through either a shunt or a series tuning capacitor.

The three Eqs. (1–3) may be reduced to only two equations since it is possible to find, by means of a reasonable approximation, a relation linking together the laws of motion of the piston, displacer, and casing, namely, $x_p(t)$, $x_d(t)$, and $x_c(t)$. To this end, let us consider the equation of motion of the c.m. of the entire machine:

$$M_p\ddot{x}_p + M_d\ddot{x}_d + M_c\ddot{x}_c + S_c x_c = 0 \tag{8}$$

that can be obtained also by summing both sides of Eqs. (1–3). The force exerted by S_c is small compared to the inertia force, so that the whole engine can be treated as a free-body.[8] Thus, Eq. (8) may be written in the form:

$$\frac{d}{dt}(M_p\dot{x}_p + M_d\dot{x}_d + M_c\dot{x}_c) = 0$$

showing that the entire engine momentum is constant, and in particular is zero, because the moving elements are initially at rest. Then, the following relation is obtained:

$$x_c = \alpha_{pc}x_p - \alpha_{dc}x_d \tag{9}$$

Substituting Eq. (9) into Eqs. (1) and (2), we get a system of only two differential equations that, bearing in mind Eqs. (4–6) too, may be written as follows:

$$M_p\ddot{x}_p = (p_w - p_b)A_p - \Delta p_w A_p/2 + F_{a-sl,1} - S_{pc}[(1 + \alpha_{pc})x_p + \alpha_{dc}x_d] \tag{10}$$

$$M_d\ddot{x}_d = \Delta p_w A_d - S_{dc}[(1 + \alpha_{dc})x_d + \alpha_{pc}x_p] \tag{11}$$

where the following functional dependences are valid:

$$p_w = p_w(x_p, x_d)$$

$$p_b = p_b(x_p, x_d)$$

$$\Delta p_w = \Delta p_w(\dot{x}_p, \dot{x}_d)$$

$$F_{a-sl,1} = F_{a-sl,1}(x_p, \dot{x}_p, x_d, \dot{x}_d)$$

since the casing motion x_c is taken into account by means of Eq. (9). By solving Eqs. (10) and (11), the piston and displacer laws of motion may be obtained. From those, Eq. (9) allows the casing law of motion to be calculated. So the TMG, although having three degrees of freedom, can be analytically described, from a dynamic point of view, as a system with two degrees of freedom.

By applying the linearization procedure described,[1] Eqs. (10) and (11) can be written, respectively, in the form:

$$\ddot{x}_p + D'_{pp}\dot{x}_p + D'_{pd}\dot{x}_d + S'_{pp}x_p + S'_{pd}x_d = 0 \tag{12}$$

$$\ddot{x}_d + D'_{dp}\dot{x}_p + D'_{dd}\dot{x}_d + S'_{dp}x_p + S'_{dd}x_d = 0 \tag{13}$$

The S' and D' coefficients per unit of mass of the TMG are given in Appendix A. For example, coefficient S'_{pd} accounts for the influence on the piston motions by virtue of a spring coupling to the motions of the displacer, whereas S'_{pp} is a spring effect unique to the piston. Similarly, coefficient D'_{dp} accounts for the influence on the displacer motions by virtue of a dashpot coupling to the motions of the piston, whereas D'_{dd} is a dashpot effect unique to the displacer.

If the machine is working in some steady-state operation, then the displacer and piston motions are sinusoidal according to the adopted linearization technique[1]

$$x_p(t) = (X_p/2)\sin(\omega t - \phi) \tag{14}$$

$$x_d(t) = (rX_p/2)\sin \omega t \tag{15}$$

where the four unknown variables X_p, r, ω, and ϕ may be obtained by solving the four nonlinear algebraic equations reported in Ref. 1. Consequently, also the casing motions, given by Eq. (9), will be sinusoidal:

$$x_c(t) = (X_c/2)\sin(\omega t - \phi_c) \tag{16}$$

where

$$X_c = X_p\sqrt{\alpha_{pc}^2 + 2\alpha_{pc}\alpha_{dc}r\cos\phi + (r\alpha_{dc})^2}$$

$$\tan\phi_c = \frac{\alpha_{pc}\sin\phi}{\alpha_{pc}\cos\phi + \alpha_{dc}r}$$

TMG Thermodynamic Analysis

The algebraic expressions of the S'_w and D'_w coefficients, defined in Appendix A, depend on the thermodynamic model selected for the working gas circuit of the machine. If an isothermal model is assumed, it is a simple matter to calculate the partials of p_w in $(x_p, x_d) = (0, 0)$, which are involved in the S'_w coefficients, and the $Y_{fl,eq}$ term that appears in the D'_w coefficients (the report[9] may be helpful in order to calculate $Y_{fl,eq}$). Nevertheless, the isothermal model does not lead to quite satisfactory results, as will be shown later. The adiabatic model, on the other hand, cannot be applied to the TMG, as already stated.

Therefore, a different thermodynamic model has to be introduced for this engine, which does not have a heater and cooler, in order to better describe the behavior of its working spaces, which have to exchange not only work, but also heat with the external environment.

The so-called polytropic model, here developed, assumes for the working gas circuit the temperature profile shown in Fig. 2. The regenerator is assumed to be ideal and the gas temperatures in the working spaces are assumed to be time-dependent, as in the adiabatic model. The mean effective regenerator temperature can be expressed as a logarithmic mean of T_h and T_k,[4] T_h and T_k being the regenerator outlet gas temperatures. The interface temperatures, according to the concept of conditional temperatures introduced by Finkelstein,[10] are

$$T_{c/rg} = \begin{cases} T_k & \text{if } \dot{m}_{c/rg} < 0 \\ T_c(x_p, x_d) & \text{if } \dot{m}_{c/rg} > 0 \end{cases}$$

$$T_{rg/e} = \begin{cases} T_h & \text{if } \dot{m}_{rg/e} > 0 \\ T_e(x_p, x_d) & \text{if } \dot{m}_{rg/e} < 0 \end{cases}$$

Since the functional dependences of T_c and T_e on x_p and x_d are unknown, the following relation:

$$p_w = m_w R/(V_c/T_c + V_{rg}/T_{rg} + V_e/T_e)$$

with

$$V_e = V_{e,m} + A_d[(1 + \alpha_{dc})x_d + \alpha_{pc}x_p]$$

$$V_c = V_{c,m} + A_p x_p - A_d x_d$$

does not allow the partials of p_w in (0, 0), which appear in the S'_w coefficients, to be calculated immediately.

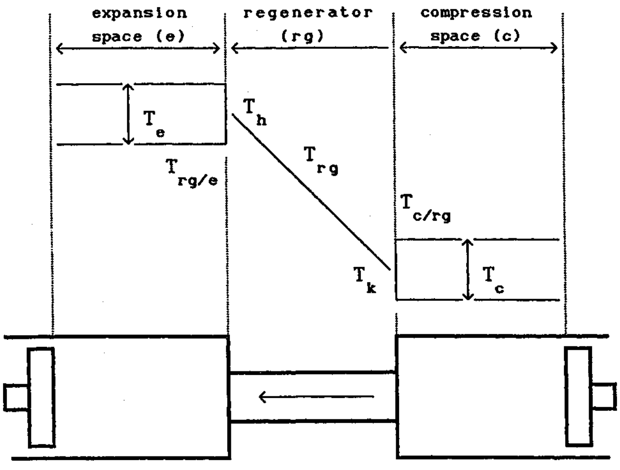

Fig. 2 Temperature profile in the TMG working gas circuit.

However, this calculation can be carried out as follows. The starting point is that the gas mass m_w in the working fluid circuit, given by the following expression:

$$m_c + m_{rg} + m_e = m_w \tag{17}$$

is constant when a cyclic steady-state operation is established. In fact, the value of m_w changes only when the operating conditions change (see subsection of Appendix A). Thus, differentiating Eq. (17) with respect to t:

$$\frac{d}{dt}(m_c + m_{rg} + m_e) = 0 \tag{18}$$

By applying the ideal gas law to the regenerator, we get

$$\frac{dm_{rg}}{dt} = \frac{V_{rg}}{T_{rg}R}\frac{dp_w}{dt} \tag{19}$$

Instead, by applying the energy equation linked with the continuity equation to the working spaces, we get

$$\frac{dm_c}{dt} = \frac{p_w}{RT_{c/rg}}\frac{dV_c}{dt} + \frac{(V_c/\gamma)}{RT_{c/rg}}\frac{dp_w}{dt} - \frac{1}{c_p T_{c/rg}}\frac{dQ_c}{dt} \tag{20}$$

$$\frac{dm_e}{dt} = \frac{p_w}{RT_{rg/e}}\frac{dV_e}{dt} + \frac{(V_e/\gamma)}{RT_{rg/e}}\frac{dp_w}{dt} - \frac{1}{c_p T_{rg/e}}\frac{dQ_e}{dt} \tag{21}$$

Substituting Eqs. (19–21) into Eq. (18), we obtain an expression for the derivative of p_w with respect to t, which is valid for irreversible processes in the compression and expansion spaces, but does not allow the first-order partials of p_w in $(x_p, x_d) = (0, 0)$ to be calculated, because of the presence of Q_c and Q_e.

Therefore, the derivatives of m_c and m_e with respect to t have to be evaluated in a different way. In case of a reversible process in the compression and expansion spaces, this evaluation can be done by applying the polytropic gas law to these spaces:

$$p_w(V_c/m_c)^{\mu_c} = c_c \qquad p_w(V_e/m_e)^{\mu_e} = c_e$$

Differentiating with respect to t

$$\frac{dm_c}{dt} = \frac{p_w}{RT_c}\frac{dV_c}{dt} + \frac{(V_c/\mu_c)}{RT_c}\frac{dp_w}{dt} \tag{22}$$

$$\frac{dm_e}{dt} = \frac{p_w}{RT_e}\frac{dV_e}{dt} + \frac{(V_e/\mu_e)}{RT_e}\frac{dp_w}{dt} \tag{23}$$

Substituting Eqs. (19), (22), and (23) into Eq. (18), and remembering the expressions of V_c and V_e, we get:

$$\frac{dp_w}{dt} = \left\{\frac{-p_w[A_p/T_c + (A_d/T_e)\alpha_{pc}]}{V_c/\mu_c T_c) + V_{rg}/T_{rg} + V_e/(\mu_e T_e)}\right\}\dot{x}_p + \left\{\frac{-p_w A_d[(1+\alpha_{dc})/T_e - 1/T_c]}{V_c/(\mu_c T_c) + V_{rg}/T_{rg} + V_e/(\mu_e T_e)}\right\}\dot{x}_d \tag{24}$$

On the other hand, the derivative of $p_w(x_p, x_d)$ with respect to t is

$$\frac{dp_w}{dt} = \left(\frac{\partial p_w}{\partial x_p}\right)\dot{x}_p + \left(\frac{\partial p_w}{\partial x_d}\right)\dot{x}_d$$

Comparing the last two equations yields the first-order partials of p_w which, calculated in $(x_p, x_d) = (0, 0)$, become

$$\left(\frac{\partial p_w}{\partial x_p}\right)_{0,0} = -\frac{p_w(0,0)}{\Gamma_{pol,\mu}}\left[\frac{A_p}{T_c(0,0)} + \frac{A_d\alpha_{pc}}{T_e(0,0)}\right]$$

$$\left(\frac{\partial p_w}{\partial x_d}\right)_{0,0} = -\frac{p_w(0,0)}{\Gamma_{pol,\mu}}\left[\frac{A_d(1+\alpha_{dc})}{T_e(0,0)} - \frac{A_d}{T_c(0,0)}\right]$$

where

$$p_w(0,0) = m_w R/\Gamma_{pol}$$

$$\Gamma_{pol} = \frac{V_{c,m}}{T_c(0,0)} + \frac{V_{rg}}{T_{rg}} + \frac{V_{e,m}}{T_e(0,0)}$$

$$\Gamma_{pol,\mu} = \frac{V_{c,m}}{\mu_c T_c(0,0)} + \frac{V_{rg}}{T_{rg}} + \frac{V_{e,m}}{\mu_e T_e(0,0)}$$

Therefore, the calculation of the first-order partials of p_w in (0, 0) requires the calculation of m_w, $T_c(0, 0)$, and $T_e(0, 0)$. The m_w may be determined by calculating the even-order partials of p_w in (0, 0), which appear in the term a_w defined in the subsection of Appendix A. The $T_c(0, 0)$ and $T_e(0, 0)$ temperatures may be evaluated as indicated in the following subsection.

The previously mentioned even-order partials of p_w in (0, 0), as well as the uneven-order partials of p_w in (0, 0), which appear in the S'_w coefficients through the term b_w, can be obtained by applying the same procedure used by the authors in Ref. 6, Chap. 5.

As far as the D'_w coefficients are concerned, it is easy to calculate, also in case of polytropic model, the $\Upsilon_{fl,eq}$ term that appears in these coefficients (the report[9] may be helpful in order to calculate $\Upsilon_{fl,eq}$).

Calculation of the Temperatures $T_c(0, 0)$ and $T_e(0, 0)$

Let us consider the functions $T_c(x_p, x_d)$ and $T_e(x_p, x_d)$ expanded in a MacLaurin series. If a cyclic steady-state working is established, the motions of piston and displacer will be periodic and stationary, and in particular, sinusoidal according to the adopted linearization procedure. Thus, neglecting the even-order nonlinear terms associated with T_c and T_e, the mean values of T_c and T_e are

$$T_{c,m} = T_c(0,0) \qquad T_{e,m} = T_e(0,0)$$

Obviously, the uneven-order nonlinear terms associated with T_c and T_e do not give any contribution to $T_{c,m}$ and $T_{e,m}$, respectively. These temperatures may be expressed in an approximate manner as follows[11]:

$$T_{c,m} = T_k + 0.5\Delta T_c \qquad T_{e,m} = T_h - 0.5\Delta T_e$$

where the maximum changes ΔT_c and ΔT_e of the gas temperatures in the compression and expansion spaces may be evaluated starting from the following equations:

$$\frac{1}{T_c}\frac{dT_c}{dt} = \frac{1}{p_w}\left(\frac{\mu_c - 1}{\mu_c}\right)\frac{dp_w}{dt}$$

$$\frac{1}{T_e}\frac{dT_e}{dt} = \frac{1}{p_w}\left(\frac{\mu_e - 1}{\mu_e}\right)\frac{dp_w}{dt}$$

In the previous equations, obtained by applying the ideal gas law to m_c and m_e appearing, respectively, in Eqs. (22) and (23), the derivative of p_w with respect to time is furnished by Eq. (24). With suitable manipulations, we were able to obtain

$$\frac{\Delta T_c}{T_{c,m}} = \left(\frac{X_p}{2}\right)\sigma_c \qquad \frac{\Delta T_e}{T_{e,m}} = \left(\frac{X_p}{2}\right)\sigma_e \tag{25}$$

where σ_c and σ_e are defined in Appendix B.

Evaluation of μ_c and μ_e

The exponents μ_c and μ_e of the polytropic processes fall into the range $[1, \gamma]$. Their values can be determined in such a way that the theoretical prediction closely matches the experimental result.

Obviously, the values obtained for the exponents μ_c and μ_e depend not only on the working machine, but also on its operating conditions.

Heat Exchanged and Thermal Efficiency

The mean thermal powers $P_{Q,c}$ and $P_{Q,e}$ respectively, rejected by the compression space and supplied to the expansion space are

$$P_{Q,c} = f\oint \frac{dQ_c}{dt}\,dt \qquad P_{Q,e} = f\oint \frac{dQ_e}{dt}\,dt$$

where the derivatives of Q_c and Q_e with respect to t can be calculated by means of Eqs. (20) and (21). However, these equations are valid for irreversible processes in the working spaces.

Since now we are dealing with reversible processes in these spaces, we can still use Eqs. (20) and (21), but with the condition

$$T_{c/rg} = T_c \qquad T_{rg/e} = T_e$$

which ensures the reversibility of the transformations in the compression and expansion spaces. Then in the reversible case Eq. (20) may be written as follows:

$$T_c\frac{dm_c}{dt} = \frac{p_w}{R}\frac{dV_c}{dt} + \frac{V_c}{R\gamma}\frac{dp_w}{dt} - \frac{1}{c_p}\frac{dQ_c}{dt} \tag{26}$$

Equation (26) does not allow $P_{Q,c}$ to be determined analytically, because we are not able to calculate the cyclic integration of the left-hand term. But in the reversible case Eq. (22) is valid too. Therefore, comparing Eqs. (22) and (26) yields the following result:

$$\frac{dQ_c}{dt} = -\frac{\gamma - \mu_c}{\mu_c(\gamma - 1)}V_c\frac{dp_w}{dt} \tag{27}$$

In the case considered so far for a TMG modeled by a linearization procedure concerning p_w, p_b, and Δp_w, the piston, displacer, and casing motions are sinusoidal when the machine is working in some steady-state operation, as shown by Eqs. (14–16). So, the cyclic integration of Eq. (27), multiplied by $f = \omega/2\pi$, gives

$$P_{Q,c} = -\frac{\gamma - \mu_c}{2\mu_c(\gamma - 1)}\,\omega\left(\frac{X_p}{2}\right)^2 r\sin\phi\, b_w\frac{m_w R}{\Gamma_{pol}\Gamma_{pol,\mu}} \times \frac{A_d}{T_{e,m}}[A_p(1 + \alpha_{dc}) + A_d\alpha_{pc}]$$

Similarly, for the expansion space

$$P_{Q,e} = \frac{\gamma - \mu_e}{2\mu_e(\gamma - 1)}\,\omega\left(\frac{X_p}{2}\right)^2 r\sin\phi\, b_w\frac{m_w R}{\Gamma_{pol}\Gamma_{pol,\mu}} \times \frac{A_d}{T_{c,m}}[A_p(1 + \alpha_{dc}) + A_d\alpha_{pc}]$$

Then, the thermal efficiency is given by

$$\eta_t = 1 - \frac{T_{c,m}}{T_{e,m}}\left[\frac{\mu_e(\gamma - \mu_c)}{\mu_c(\gamma - \mu_e)}\right]$$

The previous expressions have been obtained in the hypothesis of unitary effectiveness of the regenerator. Since the actual regeneration is imperfect, suitable corrections have to be introduced, e.g., by applying the Simple Analysis proposed by Urieli and Berchowitz in Ref. 4, Chap. 6.

Electric Power Delivered by the TMG

The linear alternator used in the TMG tested by Cooke-Yarborough for the authors' use is connected through a series tuning capacitor to a static, resistive load. Therefore, the electric circuit equivalent to the TMG linear alternator–static load subsystem is a one-loop circuit, as shown in Ref. 1.

The relations obtained[1] for the calculation of the amplitude I_g of the electric current induced in the alternator coils and of the delivered mean useful power P_u, in case of FPSE/LA–sL systems, cannot be applied to the TMG directly, because in this engine the casing motion cannot be neglected. Therefore, in this section, the right expressions for the calculation of the above quantities are given.

The electric current amplitude I_g may be obtained from Eq. (7), bearing in mind Eqs. (9), (14), and (15). With suitable manipulations, we get

$$I_g = (\omega X_p/2)\sqrt{[(a'_{1,1}/\omega)^2 + (a'_{2,1})^2]\mathcal{F}} \tag{28}$$

where $\mathcal{F}$ is given by

$$\mathcal{F} = (1 + \alpha_{pc})^2 + 2(1 + \alpha_{pc})\alpha_{dc}r\cos\phi + (\alpha_{dc}r)^2$$

The $\mathcal{F}$ coefficient is greater than 1, although it is approximately equal to 1 since in the TMG usually $M_c >> M_p$ and $M_c >> M_d$, such that α_{pc} and α_{dc} approach 0. The P_{el} delivered by the TMG with regard to R_l is

$$P_{el} = \tfrac{1}{2}R_l I_g^2 \tag{29}$$

while the mean useful power P_u delivered by the TMG to the engine/alternator interface may be obtained by solving the following cyclic integral:

$$P_u = f\oint(-F_{a-sl,1})(\dot{x}_p - \dot{x}_c)\,dt$$

Bearing in mind Eqs. (5), (7), (9), (14), and (15), with suitable manipulations, we get

$$P_u = \tfrac{1}{2}k_g a'_{2,1}(\omega X_p/2)^2\mathcal{F} \tag{30}$$

The previous expressions of I_g and P_u differ from those obtained in Ref. 1, only because of the presence of the correction factor $\mathcal{F}$, which takes into account the casing motion.

Comparison of TMG Experimental and Calculated Results

The engine tested by Cooke-Yarborough is heated by propane, and delivers about 130 W, with an efficiency of about 14% at a hot-end temperature of just under 600°C.

Normally, the engine charges a 24-V battery, but during the test procedure Cooke-Yarborough replaced, for our purpose, the battery with a variable resistor, R_l. The instrumentation measured the current flowing into this load. A small computer interfaced to the instrumentation of this engine recorded selected data continuously.

The burner is temperature-stabilized by automatically switching it on and off, so that the temperature rises and falls by about 30°C. At 587°C the burner cuts-out and the temperature falls gradually to 554°C, when the burner relights. No readings were taken while the burner was off. All the readings were taken while the temperature was rising, so that the curves are steeper than they would be with a constant hot-end temperature T_h.[5]

During the test, R_l was varied in nonuniform increments from 3.46 Ω up to 5.53 Ω, as indicated in Table 1, and a resonance condition inside the electric circuit equivalent to the linear alternator–static load subsystem was established. Thus, the $a'_{1,1}$ coefficient, which appears in Eqs. (7) and (28), is equal to zero, while the $a'_{2,1}$ coefficient, which appears in Eqs. (7), (28), and (30), assumes a simplified expression.[1]

Since the engine hot-end temperature T_h was changing all the time during the test, it was difficult to interpret the experimental results. Then, in order to obtain the theoretical results, we had to fix a correspondence between the R_l and T_h values, as shown in Table 1. This correspondence, affected by an error due to the uncertainty of T_h, is however, justified by the adopted test procedure, in accordance with which the T_h temperature rises into the range (554–587°C).

The theoretical results have been obtained by solving the four nonlinear algebraic equations, reported in Ref. 1, in the four unknown variables X_p, r, ω, and ϕ. Then, Eqs. (28) and (29) allow I_g and P_{el} to be calculated. Since the stiffness S'_w and damping D'_w coefficients per unit element mass depend on the thermodynamic model selected for the TMG working gas circuit, the theoretical analysis has been performed assuming both an isothermal and a polytropic model.

The calculation has been carried out by changing the load resistance values in accordance with the described test procedure, and assuming the correspondence of Table 1 is valid. In particular, we have calculated, for the considered TMG, the theoretical curves of the following variables: piston amplitude $X_p/2$, displacer amplitude $X_d/2$, electric current amplitude I_g, ϕ, f, and P_{el}, as functions of R_l. For the tested TMG, using helium as a working fluid ($\gamma = 1.67$), we have found that

$$(\mu_c, \mu_e) = (1.35, 1.15)$$

when $R_l = 3.46$ Ω. The variation of μ_c and μ_e is less than or equal to 5% when the load resistance changes. Figures 3–8 show the comparison of the experimental (○) and calculated (polytropic: △; isothermal: □) curves for the previous variables. It is interesting to note that the output current amplitude, as well as the mean electric power, rise progressively with the load resistance. This happens, as already explained[1] if R_l is less than a certain value, otherwise the amplitude I_g decreases with the resistive load. This critical value of R_l, however, is not included in the range considered by Cooke-Yarborough.

Table 2 gives the minimum and maximum percent (%) differences between the calculated and experimental results.

Table 1 Correspondence between the R_l and T_h values

R_l, Ω	T_h, °C	R_l, Ω	T_h, °C
3.46	554	4.88	554
3.67	565	5.25	570.5
4.11	576	5.53	587
4.48	587	—	—

Table 2 Percent (%) differences between the TMG theoretical and experimental performance

	X_p	X_d	I_g	ϕ	P_{el}
Polytropic	6 → 12	7 → 13	6 → 10	12 → 17	15 → 22
Isothermal	12 → 18	13 → 20	8 → 12	15 → 23	21 → 30

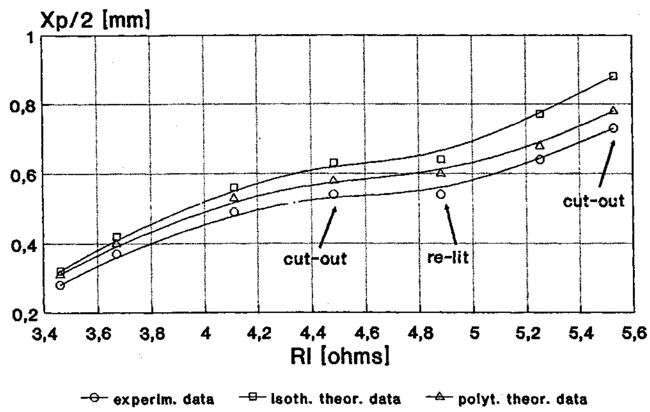

Fig. 3 Piston amplitude as a function of the load resistance.

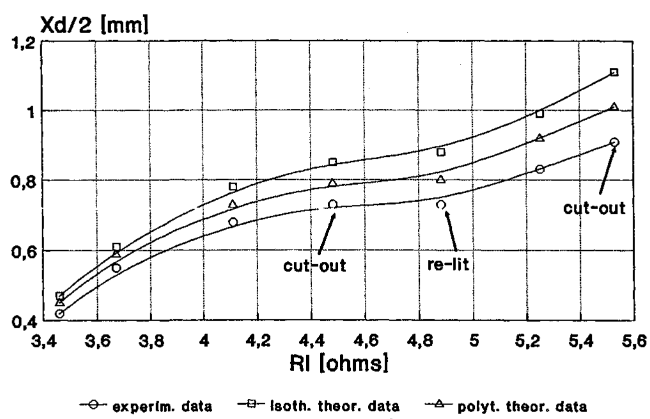

Fig. 4 Displacer amplitude as a function of the load resistance.

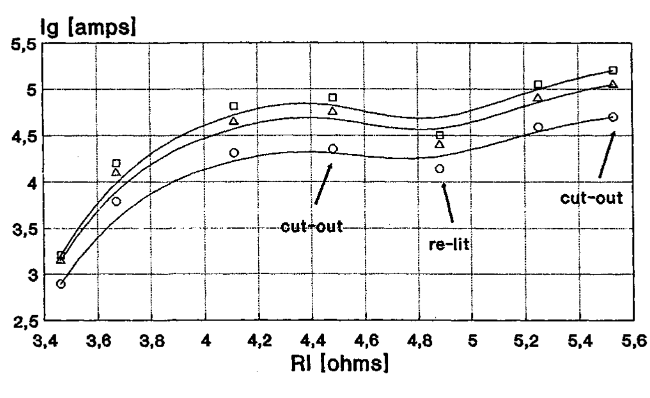

Fig. 5 Electric current amplitude as a function of the load resistance.

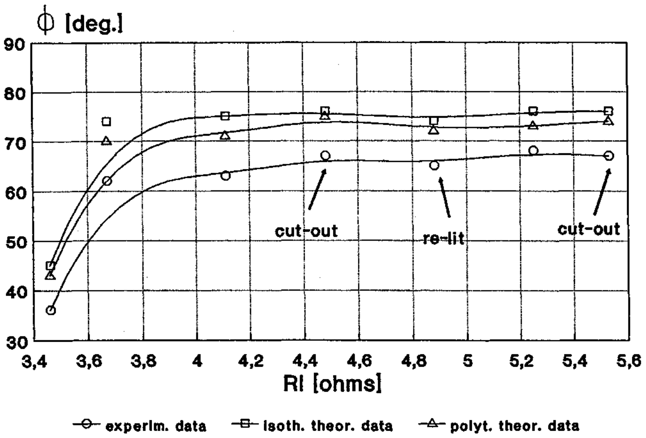

Fig. 6 Piston–displacer phase angle as a function of the load resistance.

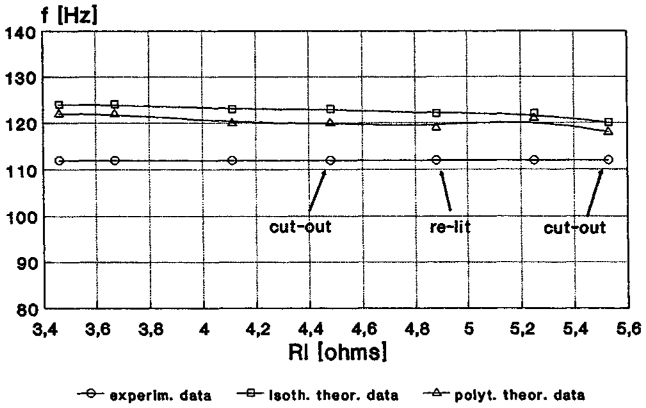

Fig. 7 Frequency as a function of the load resistance.

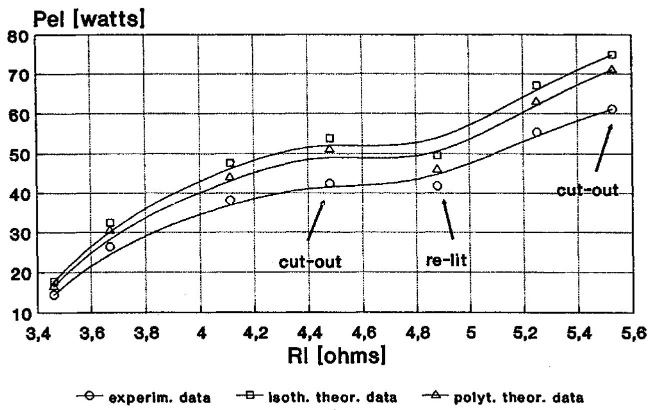

Fig. 8 Electric power as a function of the load resistance.

In the case of polytropic (nonisothermal) working spaces the calculated and experimental values for the TMG differ by not more than 22% for all parameters.

As shown in Figs. 3–8 and summarized in Table 2, there is a substantial agreement between the experimental data and the theoretical prediction. This agreement is especially true in a relative sense, i.e., if one considers the results obtained by the most reliable theoretical methods developed for the performance prediction of Stirling engines and, in particular, of FPSEs.

The deviations between the test data and the results obtained with the present analysis, in spite of experimental uncertainties and theoretical limitations, are less than those shown in previous studies on the TMG performance (Ref. 7, Chap. 6 and Ref. 12), where in addition the comparison between theory and experiments was restricted to only one cyclic steady-state operation of the machine.

In order to further improve the accuracy of the developed analysis, it is necessary to remove some of the simplifying assumptions for the TMG, as summarized next.

Working gas circuit:

1) The working spaces are either isothermal or polytropic.

2) The temperature profile through the regenerator is linear.

3) The ideal gas equation applies.

4) The nonlinearities associated with p_w and Δp_w are represented by equivalent linear terms.

5) The nonlinearities associated with p_w of an order higher than the third one are neglected.

6) The kinetic and potential energies of the gas flow are neglected.

7) The shuttle effect is neglected.

Bounce space:

8) The nonlinearities associated with p_b are neglected.

9) The hysteresis losses are neglected.

Mechanical springs:

10) All mechanical springs are linear and massless.

11) The force exerted by the mounting springs S_c is neglected.

Linear alternator–static load subsystem:

12) The parasitic current and the hysteresis losses within the alternator magnetic body are neglected.

13) The magnetic reluctance of the alternator iron is neglected.

Centering device:

14) The centering port for the piston and casing has an ideal operation, as explained in a subsection of Appendix A.

Effect of the Stabilizing Elements on the TMG Operation

In past studies we have singled out all the elements which have a stabilizing effect on the operation of an FPSE connected to a generic load device–load subsystem. They are summarized in Ref. 1. In the case of the TMG, these elements are: 1) the nonlinearities associated with p_w, 2) the nonlinearity associated with Δp_w, and 3) the nonisothermal behavior of the working spaces.

The LA–sL subsystem does not affect the machine stability.[1]

Figure 9, obtained by the proposed methodology via considering separately the various stabilizing elements, shows the effect on the TMG stability of each element when the R_l changes. In particular, the nonlinearities associated with p_w and the nonlinearity associated with Δp_w are convenient in

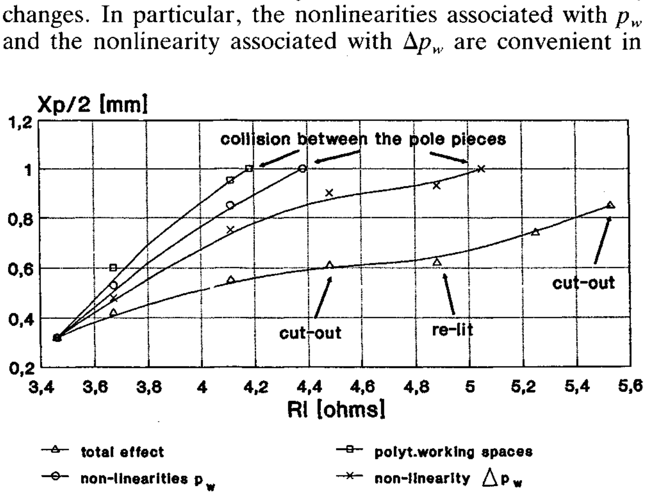

Fig. 9 Effect of the various stabilizing elements on the TMG operation.

the TMG, because they improve its stability more than the nonisothermal behavior of the working spaces. However, the nonisothermal working spaces reduce the machine efficiency.

For a different TMG the stabilizing effect of the three elements might be different. Thus, the results obtained here are rigorously valid only for the tested machine, and may be extended to other TMGs only qualitatively.

Conclusions

The substantial agreement obtained between the experimental data and the theoretical prediction of the TMG performance, in a wide range of operating conditions, validates the coupled electric, dynamic, and thermodynamic analysis developed by the authors for FPSE/LA systems and published in the first part of this study.[1]

A notable advantage of the proposed method, which is not a simulation technique, but rather a fully analytical procedure, is that it can be used not only for the performance prediction, but also for the design of FPSE/LA systems. This is true in particular for those engines whose thermodynamic behavior is described by either an isothermal or an adiabatic model.

However, in the case considered here for the TMG, if the new polytropic model is assumed, since the adiabatic one cannot be used, the mathematical modeling does not account for the engine physics to the extent of extrapolating other geometries. In other words, the polytropic exponents μ_c and μ_e must be adjusted for engines of different geometries. Hence, the proposed coupled analysis may be used for the design of a TMG, only if it is linked with an isothermal model and, therefore, only for a preliminary design of the machine.

On the other hand, since the developed method allows the effect of the different stabilizing elements to be quantified separately, it can be used, during the machine design, in order to ensure not only a steady-state operation, but above all, a quite stable operation, especially when large variations of the operating conditions are required.

Appendix A: TMG Coefficients

The stiffness S' and damping D' coefficients of the TMG system, which appear in Eqs. (12) and (13), may be decomposed into the sum of four coefficients:

S' stiffness coefficients per unit of mass

$$S'_{pp} = S'_{pp,w} + S'_{pp,\text{ms}} + S'_{pp,b} + S'_{pp,a-\text{sl}}$$

$$S'_{pd} = S'_{pd,w} + S'_{pd,\text{ms}} + S'_{pd,b} + S'_{pd,a-\text{sl}}$$

$$S'_{dp} = S'_{dp,w} + S'_{dp,\text{ms}} + S'_{dp,b} + S'_{dp,a-\text{sl}}$$

$$S'_{dd} = S'_{dd,w} + S'_{dd,\text{ms}} + S'_{dd,b} + S'_{dd,a-\text{sl}}$$

D' damping coefficients per unit of mass

$$D'_{pp} = D'_{pp,w} + D'_{pp,\text{ms}} + D'_{pp,b} + D'_{pp,a-\text{sl}}$$

$$D'_{pd} = D'_{pd,w} + D'_{pd,\text{ms}} + D'_{pd,b} + D'_{pd,a-\text{sl}}$$

$$D'_{dp} = D'_{dp,w} + D'_{dp,\text{ms}} + D'_{dp,b} + D'_{dp,a-\text{sl}}$$

$$D'_{dd} = D'_{dd,w} + D'_{dd,\text{ms}} + D'_{dd,b} + D'_{dd,a-\text{sl}}$$

The right-hand coefficients are given separately next.

Working Gas Circuit

Stiffness coefficients per unit of mass due to p_w

$$S'_{pp,w} = -b_w \frac{A_p}{M_p}\left(\frac{\partial p_w}{\partial x_p}\right)_{0,0}$$

$$S'_{pd,w} = -b_w \frac{A_p}{M_p}\left(\frac{\partial p_w}{\partial x_d}\right)_{0,0}$$

$$S'_{dp,w} = S'_{dd,w} = 0$$

The b_w coefficient takes into account the uneven-order nonlinearities associated with p_w (see Ref. 1).

Damping coefficients per unit of mass due to Δp_w

$$D'_{pp,w} = \frac{D^*_{flp}A_p/2}{M_p} \qquad D'_{pd,w} = -\frac{D^*_{fld}A_p/2}{M_p}$$

$$D'_{dd,w} = \frac{D^*_{fld}A_d}{M_d} \qquad D'_{dp,w} = -\frac{D^*_{flp}A_d}{M_d}$$

where

$$D^*_{flp} = (\tfrac{1}{2})\Upsilon_{fl,\text{eq}}[A_p - (2A_d - A_p)\alpha_{\text{pc}}]$$

$$D^*_{fld} = (\tfrac{1}{2})\Upsilon_{fl,\text{eq}}[2A_d + (2A_d - A_p)\alpha_{\text{dc}}]$$

The $\Upsilon_{fl,\text{eq}}$ coefficient represents the linearized pressure drop per unit of volumetric flow rate through the regenerator.[1]

Mechanical Springs

Stiffness coefficients per unit of mass

$$S'_{pp,\text{ms}} = \frac{S_{\text{pc}}}{M_p}(1 + \alpha_{\text{pc}}) \qquad S'_{pd,\text{ms}} = \frac{S_{\text{pc}}}{M_p}\alpha_{\text{dc}}$$

$$S'_{dd,\text{ms}} = \frac{S_{\text{dc}}}{M_d}(1 + \alpha_{\text{dc}}) \qquad S'_{dp,\text{ms}} = \frac{S_{\text{dc}}}{M_d}\alpha_{\text{pc}}$$

Damping coefficients per unit of mass

$$D'_{pp,\text{ms}} = D'_{pd,\text{ms}} = D'_{dp,\text{ms}} = D'_{dd,\text{ms}} = 0$$

Bounce Space

Stiffness coefficients per unit of mass due to p_b

$$S'_{pp,b} = b_b \frac{A_p}{M_p}\left(\frac{\partial p_b}{\partial x_p}\right)_{0,0}$$

$$S'_{pd,b} = b_b \frac{A_p}{M_p}\left(\frac{\partial p_b}{\partial x_d}\right)_{0,0}$$

$$S'_{dp,b} = S'_{dd,b} = 0$$

Damping coefficients per unit of mass

$$D'_{pp,b} = D'_{pd,b} = D'_{dp,b} = D'_{dd,b} = 0$$

The algebraic expressions of the S'_b coefficients may be evaluated assuming an ideal isothermal behavior for the bounce space, as explained in Appendix A of Ref. 1.

LA–sL Subsystem

Stiffness coefficients per unit of mass

$$S'_{pp,a-\text{sl}} = (k_g a'_{1,1}/M_p)(1 + \alpha_{\text{pc}})$$

$$S'_{pd,a-\text{sl}} = (k_g a'_{1,1}/M_p)\alpha_{\text{dc}}$$

$$S'_{dp,a-\text{sl}} = S'_{dd,a-\text{sl}} = 0$$

Damping coefficients per unit of mass

$$D'_{pp,a-sl} = (k_g a'_{2,1}/M_p)(1 + \alpha_{pc})$$

$$D'_{pd,a-sl} = (k_g a'_{2,1}/M_p)\alpha_{dc}$$

$$D'_{dp,a-sl} = D_{dd,a-sl} = 0$$

where the terms $a'_{1,1}$ and $a'_{2,1}$, as is already stated, are given in Ref. 1. The k_g positive constant, called the alternator constant, assumes the following expression (Ref. 7, Chap. 6):

$$k_g = BA_{pm}N/(2w)$$

Centering Devices

Since the TMG does not have the gas spring typical of other FPSEs, the relative seal leakage losses are obviously absent. Therefore, the midpoint of the displacer stroke holds a fixed location during the engine cyclic steady-state operation. Since this location depends only on M_d and S_{dc}, it does not change even when the operating conditions T_h, T_k, and R_l are varied. Therefore, the mean volume $V_{e,m}$ does not change.

Instead, the midpoints of the piston and casing strokes hold a fixed location when the engine is working in some steady-state operation, because seal leakage losses between the compression space and the bounce space are, as is well known, absent in the TMG, while they tend to move when the operating conditions are varied. In fact, any change in the operating conditions will modify the mean pressure of the working gas circuit so displacing the mean position of the piston (diaphragm) as well as the mean position of the casing. In order to avoid this effect, in the TMG a centering port is used. It operates on the gas masses m_w and m_b in the respective spaces by means of deliberately introduced leakage: the gas slowly leaks through a small hole of the diaphragm hub to equalize the mean pressures on the two sides of both the diaphragm and the casing.

Although this leakage cannot completely equalize the mean pressures, the proposed analysis assumes an ideal behavior for the centering port. This implies that, when the operating conditions are varied, the gas masses m_w and m_b change, but the mean volumes $V_{c,m}$ and $V_{b,m}$ do not change.

The values of the gas masses are given by

$$p_{w,m} = p_{b,m}$$

$$m_w + m_b = m_{tot}$$

m_{tot} being the total mass of gas within the machine. It may be proved that the m_w appears in the stiffness coefficients S'_w per unit element mass by means of the first-order partials of p_w calculated in (0, 0), independently of the thermodynamic model selected for the working gas circuit. Similarly, m_b appears in the S'_b coefficients.

The mean pressures are given by

$$p_{w,m} = a_w p_w(0, 0)$$

$$p_{b,m} = p_b(0, 0)$$

where the a_w coefficient takes into account the even-order nonlinearities associated with p_w (see Ref. 1). The expressions of $p_{w,m}$ and $p_{b,m}$ depend on the thermodynamic model assumed for the respective spaces.

Appendix B: σ_c and σ_e Expressions

The variables σ_c and σ_e in Eqs. (25) are given by

$$\sigma = \left\{\left(\frac{A_p}{T_k} + \frac{A_d\alpha_{pc}}{T_h}\right)^2 + \left[\frac{A_d(1+\alpha_{dc})}{T_h} - \frac{A_d}{T_k}\right]^2 r^2 + \left(\frac{A_p}{T_k} + \frac{A_d\alpha_{pc}}{T_h}\right)\left[\frac{A_d(1+\alpha_{dc})}{T_h} - \frac{A_d}{T_k}\right] \times 2r\cos\phi\right\}^{1/2} \frac{\mu - 1}{\mu\Gamma'_{pol,\mu}}$$

where $\sigma = \sigma_c$ when $\mu = \mu_c$, and $\sigma = \sigma_e$ when $\mu = \mu_e$. The $\Gamma'_{pol,\mu}$ term is given by

$$\Gamma'_{pol,\mu} = \frac{V_{c,m}}{\mu_c T_k} + \frac{V_{rg}}{T_{rg}} + \frac{V_{e,m}}{\mu_e T_h}$$

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